Int. J. Solids Structures, 1970, Vol. 6, pp. 1 to 20, Pergamon Press. Printed in Great Britain

ON LINEAR VISCOELASTIC RODS

WILLIAM J. SHACK

Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts

Abstract—This paper is based on the general thermodynamical theory of a Cosserat continuum developed by Green and Laws. We present here specific constitutive equations for a linear viscoelastic material. When the form of the free energy is restricted by certain symmetry conditions, the basic equations separate into four groups, two for flexure, one for torsion, and one for extension of the rod. Thermal effects occur only in the last group. Flexural and torsional wave propagation along an infinite rod are considered.

1. INTRODUCTION

THIS paper presents a theory of a one dimensional, viscoelastic, Cosserat continuum. Following Green and Laws [1], we shall define a rod as a one dimensional Cosserat continuum. Green and Laws have developed an exact thermodynamical theory of rods, which is not restricted to small deformations or elastic rods. Of course, a theory of rods can also be constructed by considering the rod as a three dimensional body. The equations governing the rod are then obtained from the three dimensional equations by introducing assumption or expansions based on the "thinness" of the rod. Some results which show that the Cosserat theory is a natural first approximation to the three dimensional problem have recently been given by Green, Laws and Naghdi [3]. In Section 5 of this paper, we shall briefly discuss how solutions from the Cosserat theory of rods can be compared with corresponding exact solutions from the three dimensional theory of linear viscoelasticity.

Recently Green, Laws and Naghdi [2] have used the basic theory of [1] to derive a linear theory of straight elastic rods, although as the authors note their work could be readily extended to the case where the rod has an initial curvature.

In this paper, using a procedure similar to that of [2], we derive a linear theory of viscoelastic rods which are initially straight. The basic field equations of the theory, which are the same as those given in [2], are presented in Section 2. In Section 3 appropriate constitutive equations for viscoelastic rods are discussed. The assumed form of the free energy function and the discussion of thermodynamical restrictions presented here are analogous to that of Christensen and Naghdi [8] in their work on the general, linear, three dimensional viscoelastic solid. We then restrict our attention to the case where the rod possesses certain symmetries. The resultant equations seem to correspond to those of a rod considered as an isotropic, viscoelastic, three dimensional member with a cross section which is symmetric about its principal axes. Under this symmetry restriction the equations of the theory separate into four groups: Two governing flexure, one governing torsion, and one governing longitudinal motions. We note that the temperature is present in only the last group.

In Section 5 we use the equations of Sections 3 and 4 to consider the propagation of flexural waves in an infinite rod. The solution for an arbitrary frequency is very complicated, and explicit results are presented only for the asymptotic wave speeds, attenuation and amplitudes as the frequency becomes very large. These results generalize well known results of Hunter [4]. We also compare the solution of the problem of quasi-static pure flexure using the Cosserat theory with the solution of the corresponding problem using the exact solution obtained from the three dimensional theory of linear viscoelasticity.

Finally in Section 6 we consider the torsion equations. We introduce an additional symmetry restriction on the form of the constitutive equations. Under this restriction our Cosserat rod now seems to correspond to a right circular cylinder. We then study the propagation of torsional waves through the Cosserat rod and compare our results with those of Berry [5], who solved the corresponding three dimensional problem.

2. LINEAR THEORY OF RODS

In the basic theory of rods as developed by Green and Laws [1], a rod is defined to be a curve (embedded in a Euclidean 3-space) at each point of which there are two assigned directors. The motion of the rod at time t is then described by the equations

$$\mathbf{r} = \mathbf{r}(z, t), \qquad \mathbf{a}_\alpha = \mathbf{a}_\alpha(z, t), \tag{2.1}$$

where $\mathbf{r}$ is the position vector of the curve, $\mathbf{a}_\alpha(\alpha = 1, 2)$ are the directors and z is a convected coordinate which defines points on the curve. We also define the base vector $\mathbf{a}_3$ along the curve by

$$\mathbf{a}_3 = \frac{\partial \mathbf{r}}{\partial z}, \tag{2.2}$$

and impose the restriction

$$[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] > 0. \tag{2.3}$$

In terms of $\mathbf{a}_\alpha$ and $\mathbf{a}_3$, the basic kinematical quantities may be taken as

$$a_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j, \qquad \kappa_{ij} = \mathbf{a}_j \cdot \frac{\partial \mathbf{a}_i}{\partial z}. \qquad (2.4)\dagger$$

The initial values of the position vector and the directors are denoted by $\mathbf{R}, \mathbf{A}_1$ and $\mathbf{A}_2$. We assume that the rod is initially straight so that

$$\mathbf{R} = z\mathbf{A}_3, \qquad \mathbf{A}_i \cdot \mathbf{A}_j = \delta_{ij}, \tag{2.5}$$

where $\mathbf{A}_i$ are independent of z and δ_{ij} is the Kronecker delta. We restrict our attention to the case when the subsequent displacements of the rod are "small". To be consistent we will also assume that the changes in the thermodynamic variables such as temperature are also "small". More precisely, we assume that

$$\mathbf{r} = z\mathbf{A}_3 + \varepsilon\mathbf{u}, \qquad \mathbf{a}_i = \mathbf{A}_i + \varepsilon\mathbf{b}_i, \qquad T' = T_0 + \varepsilon T, \tag{2.6}$$

where T' is the temperature of the body with value T_0 in the initial undeformed state, and ε is a small non-dimensional parameter. All forced and assigned loads are also assumed to be of $O(\varepsilon)$. By neglecting all terms of $O(\varepsilon^2)$ and higher in the field equations we obtain the

† Latin indices take the values 1, 2 and 3, and Greek indices the values 1 and 2. Also, repeated indices imply the usual summation convention.

linearized field equations. After obtaining the approximate equations, we may formally set $\varepsilon = 1$ in the equations without loss of generality with the understanding that when the displacements, temperature changes, forces, and assigned forces are expressed in suitable non-dimensional forms they are to be considered small, i.e. $\ll 1$. Since this procedure is straightforward, only the final results are presented here. Before writing down the basic equations, however, we note that as a consequence of the linearization all components of vectors such as $\mathbf{r}$ and $\mathbf{a}_i$ are now referred to the fixed orthonormal base vectors $\mathbf{A}_i$, and hence, there is no need to distinguish between covariant and contravariant components.

It is convenient to define measures of deformation by

$$\gamma_{ij} = a_{ij} - \delta_{ij},$$

or in the linearized theory

$$\gamma_{ij} = b_{ij} + b_{ji}. \tag{2.7}$$

Also, by $(2.4)_2$ and $(2.6)_1$, we have

$$\kappa_{ij} = \frac{\partial b_{ij}}{\partial z}, \qquad b_{3i} = \frac{\partial u_i}{\partial z}, \tag{2.8}$$

where u_i are the components of the displacement vector $\mathbf{u} = u_i \mathbf{A}_i$.

The linearized basic field equations of the theory are given in [2]. Here we quote freely from [2] and also record additional results appropriate to the linear theory which will be useful later. The linearized equations of motion are

$$\frac{\partial n_i}{\partial z} + \rho f_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \tag{2.9}$$

$$\pi_{\alpha\beta} = \pi_{\beta\alpha}, \qquad \pi_{\beta 3} = n_\beta, \tag{2.10}$$

where ρ is the initial mass per unit length of the rod, n_i are force components, and f_i are the components of the assigned force per unit mass on the rod. The quantities $\pi_{\alpha i}$ are defined by

$$\pi_{\alpha i} = \rho q_{\alpha i} + \frac{\partial p_{\alpha i}}{\partial z}, \tag{2.11}$$

where $p_{\alpha i}$ are the components of the director forces, and $q_{\alpha i}$ are the sum of the assigned director force on the rod and the director inertia forces.

Equations (2.10) and (2.11) can be written in a more familiar form if we introduce vectors $\mathbf{m}$ and $\mathbf{g}$ defined by

$$\mathbf{m} = \mathbf{A}_\alpha \times \mathbf{p}_\alpha, \qquad \mathbf{g} = \mathbf{A}_\alpha \times \mathbf{q}_\alpha, \tag{2.12}$$

with components

$$\begin{aligned} m_1 &= p_{23}, & m_2 &= -p_{13}, & m_3 &= p_{12} - p_{21}, \\ g_1 &= q_{23}, & g_2 &= -q_{13}, & g_3 &= q_{12} - p_{21}. \end{aligned} \tag{2.13}$$

It is then easily seen that (2.10) and (2.11) are equivalent to

$$\begin{aligned} \frac{\partial m_1}{\partial z} - n_2 + \rho g_1 &= 0, \\ \frac{\partial m_2}{\partial z} + n_1 - \rho g_2 &= 0, \\ \frac{\partial m_3}{\partial z} + \rho g_3 &= 0, \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} \pi_{11} &= \rho q_{11} + \frac{\partial p_{11}}{\partial z}, \\ \pi_{22} &= \rho q_{22} + \frac{\partial p_{22}}{\partial z}, \\ 2\pi_{12} = 2\pi_{21} &= \rho(q_{12}+q_{21}) + \frac{\partial}{\partial z}(p_{12}+p_{21}). \end{aligned} \tag{2.15}$$

We must also introduce an explicit form for the director inertia terms. If we assume that the contribution of the directors to the kinetic energy is of the form

$$\tfrac{1}{2}\alpha_1 \dot{\mathbf{a}}_1 \cdot \dot{\mathbf{a}}_1 + \tfrac{1}{2}\alpha_2 \dot{\mathbf{a}}_2 \cdot \dot{\mathbf{a}}_2, \tag{2.16}$$

where the coefficients α_1 and α_2 are independent of t and a superposed dot denotes partial differentiation with respect to time, it follows that

$$q_{\beta i} = l_{\beta i} - \alpha_\beta \frac{\partial^2 b_{\beta i}}{\partial t^2} \qquad \text{(no sum on } \beta), \tag{2.17}$$

where $l_{\beta i}$ are the assigned director forces.

The energy equation appropriate to the linear theory is

$$-\rho \dot{A} - \rho(T'\dot{S} + \dot{T}S) + \rho r + \tfrac{1}{2}\pi_{\alpha\beta}\dot{\gamma}_{\alpha\beta} + \tfrac{1}{2}n_\beta(\dot{\gamma}_{\beta 3} + \dot{\gamma}_{3\beta}) + \tfrac{1}{2}n_3\dot{\gamma}_{33} + p_{\alpha i}\dot{\kappa}_{\alpha i} - \frac{\partial h}{\partial z} = 0, \tag{2.18}$$

where

$$\dot{\gamma}_{ij} = \frac{\partial \gamma_{ij}}{\partial t}, \qquad \dot{\kappa}_{\alpha i} = \frac{\partial \kappa_{\alpha i}}{\partial t}, \tag{2.19}$$

A is the Helmholtz free energy per unit mass, r is the heat supply per unit mass per unit time and h is the flux of heat along the rod. To complete the basic equations we postulate an entropy production inequality of the form

$$\rho T'\dot{S} - \rho r + \frac{\partial h}{\partial z} - \frac{h}{T_0}\frac{\partial T}{\partial z} \geq 0. \tag{2.20}$$

If we use (2.18) to eliminate ρr from (2.20), we obtain the reduced entropy inequality

$$-\rho(\dot{A} + \dot{T}S) + \tfrac{1}{2}\pi_{\alpha\beta}\dot{\gamma}_{\alpha\beta} + \tfrac{1}{2}n_\beta(\dot{\gamma}_{\beta 3} + \dot{\gamma}_{3\beta}) + \tfrac{1}{2}n_3\dot{\gamma}_{33} + p_{\alpha i}\dot{\kappa}_{\alpha i} - \frac{h}{T_0}\frac{\partial T}{\partial z} \geq 0. \tag{2.21}$$

3. CONSTITUTIVE EQUATIONS FOR A VISCOELASTIC ROD

To obtain a complete theory we must specify appropriate constitutive equations for the free energy A, the force vector n_i, the director forces $p_{\alpha i}$, the quantities $\pi_{\alpha\beta}$, the entropy S, and the heat flux h. Since we wish to consider the case where n_i, $p_{\alpha i}$, etc. are linear functionals of the kinematical and thermal histories, it is sufficient to consider a constitutive equation for the free energy of the form

$$
\begin{aligned}
\rho A = \rho A_0 &+ \int_{-\infty}^{t} D_{ij}(t-\tau)\dot{\gamma}_{ij}(\tau)\,\mathrm{d}\tau + \int_{-\infty}^{t} \lambda(t-\tau)\dot{T}(\tau)\,\mathrm{d}\tau \\
&+ \tfrac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t} G_{ijkl}(t-\tau, t-\eta)\dot{\gamma}_{ij}(\tau)\dot{\gamma}_{kl}(\eta)\,\mathrm{d}\tau\,\mathrm{d}\eta \\
&+ \int_{-\infty}^{t}\int_{-\infty}^{t} \Phi_{ij}(t-\tau, t-\eta)\dot{\gamma}_{ij}(\tau)\dot{T}(\tau)\,\mathrm{d}\tau\,\mathrm{d}\eta \\
&+ \tfrac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t} m(t-\tau, t-\eta)\dot{T}(\tau)\dot{T}(\eta)\,\mathrm{d}\tau\,\mathrm{d}\eta + \int_{-\infty}^{t} H_{\alpha i}(t-\tau)\dot{\kappa}_{\alpha i}\,\mathrm{d}\tau \qquad (3.1) \\
&+ \tfrac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t} H_{\alpha i\beta j}(t-\tau, t-\eta)\dot{\kappa}_{\alpha i}(\tau)\dot{\kappa}_{\beta j}(\eta)\,\mathrm{d}\tau\,\mathrm{d}\eta \\
&+ \int_{-\infty}^{t}\int_{-\infty}^{t} F_{\alpha ijk}(t-\tau, t-\eta)\dot{\kappa}_{\alpha i}(\tau)\dot{\gamma}_{jk}(\eta)\,\mathrm{d}\tau\,\mathrm{d}\eta \\
&+ \int_{-\infty}^{t}\int_{-\infty}^{t} \Gamma_{\alpha i}(t-\tau, t-\eta)\dot{\kappa}_{\alpha i}(\tau)\dot{T}(\eta)\,\mathrm{d}\tau\,\mathrm{d}\eta,
\end{aligned}
$$

where as we recall a superposed dot denotes partial differentiation with respect to the time variable τ. Without loss of generality, we may assume that

$$
\begin{aligned}
G_{ijkl}(t-\tau, t-\eta) &= G_{klij}(t-\eta, t-\tau), \\
m(t-\tau, t-\eta) &= m(t-\eta, t-\tau), \qquad (3.2) \\
H_{\alpha i\beta j}(t-\tau, t-\eta) &= H_{\beta j\alpha i}(t-\eta, t-\tau),
\end{aligned}
$$

and that the kernel functions in (3.1) vanish for negative values of their arguments.

The heat flux is assumed to be linear in the history $(\partial\dot{T}/\partial z)(\tau)$

$$
h = -\int_{-\infty}^{t} k(t-\tau)\frac{\partial \dot{T}}{\partial z}(\tau)\,\mathrm{d}\tau. \qquad (3.3)
$$

It is not necessary to record here specific constitutive equations for $n_i, p_{\alpha i}, \pi_{\alpha\beta}$ and S; it suffices to assume that they are linear integral operators over the histories of $\dot{\gamma}_{ij}, \dot{\kappa}_{\alpha i}$ and $\dot{T}$, but that they are independent of the temperature gradient.

Substituting from (3.1) into the reduced entropy inequality (2.20), we obtain

$$
\begin{aligned}
&\Big[-\rho S-\lambda(0)-\int_{-\infty}^{t} m(0,t-\tau)\dot{T}(\tau)\,\mathrm{d}\tau \\
&\quad-\int_{-\infty}^{t}\Phi_{ij}(t-\tau,0)\dot{\gamma}_{ij}(\tau)\,\mathrm{d}\tau-\int_{-\infty}^{t}\Gamma_{\alpha i}(t-\tau,0)\dot{\kappa}_{\alpha i}(\tau)\,\mathrm{d}\tau\Big]\dot{T}(t) \\
&\quad+\Big[\pi_{\alpha\beta}-2D_{\alpha\beta}(0)-2\int_{-\infty}^{t}G_{\alpha\beta kl}(0,t-\tau)\dot{\gamma}_{kl}(\tau)\,\mathrm{d}\tau \\
&\quad-2\int_{-\infty}^{t}\Phi_{\alpha\beta}(0,t-\tau)\dot{T}(\tau)\,\mathrm{d}\tau-2\int_{-\infty}^{t}F_{\gamma i\alpha\beta}(t-\tau,0)\dot{\kappa}_{\alpha i}(\tau)\,\mathrm{d}\tau\Big]\dot{\gamma}_{\alpha\beta}(t) \\
&\quad+\Big[2n_{\beta}-4\int_{-\infty}^{t}G_{\beta 3kl}(0,t-\tau)\dot{\gamma}_{kl}(\tau)\,\mathrm{d}\tau-4\int_{-\infty}^{t}\Phi_{\beta 3}(0,t-\tau)\dot{T}(\tau)\,\mathrm{d}\tau \\
&\quad-4\int_{-\infty}^{t}F_{\alpha i\beta 3}(t-\tau,0)\dot{\kappa}_{\alpha i}(\tau)\,\mathrm{d}\tau-4D_{\beta 3}(0)\Big]\dot{\gamma}_{\beta 3}(t) \\
&\quad+\Big[n_{3}-2D_{33}(0)-2\int_{-\infty}^{t}G_{33kl}(0,t-\tau)\dot{\gamma}_{kl}(\tau)\,\mathrm{d}\tau \\
&\quad-2\int_{-\infty}^{t}\Phi_{33}(0,t-\tau)\dot{T}(\tau)\,\mathrm{d}\tau-2\int_{-\infty}^{t}F_{\alpha i33}(t-\tau,0)\dot{\kappa}_{\alpha i}(\tau)\,\mathrm{d}\tau\Big]\dot{\gamma}_{33}(t) \\
&\quad+\Big[p_{\alpha i}-H_{\alpha i}(0)-\int_{-\infty}^{t}H_{\alpha i\beta j}(0,t-\tau)\dot{\kappa}_{\beta j}(\tau)\,\mathrm{d}\tau \\
&\quad-\int_{-\infty}^{t}F_{\alpha ijk}(0,t-\tau)\dot{\gamma}_{jk}(\tau)\,\mathrm{d}\tau-\int_{-\infty}^{t}\Gamma_{\alpha i}(0,t-\tau)\dot{T}(\tau)\,\mathrm{d}\tau\Big]\dot{\kappa}_{\alpha i}(t) \\
&\quad-\Lambda-\int_{-\infty}^{t}\frac{\partial}{\partial t}D_{ij}(t-\tau)\dot{\gamma}_{ij}(\tau)\,\mathrm{d}\tau-\int_{-\infty}^{t}\frac{\partial}{\partial t}\lambda(t-\tau)\dot{T}(\tau)\,\mathrm{d}\tau \\
&\quad-\int_{-\infty}^{t}\frac{\partial}{\partial t}H_{\alpha i}(t-\tau)\dot{\kappa}_{\alpha i}(\tau)\,\mathrm{d}\tau-\frac{h}{T_0}\frac{\partial T}{\partial z}(t)\geq 0,
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
\Lambda &=\frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t}\frac{\partial}{\partial t}G_{ijkl}(t-\tau,t-\eta)\dot{\gamma}_{ij}(\tau)\dot{\gamma}_{kl}(\eta)\,\mathrm{d}\tau\,\mathrm{d}\eta \\
&\quad+\int_{-\infty}^{t}\int_{-\infty}^{t}\frac{\partial}{\partial t}\Phi_{ij}(t-\tau,t-\eta)\dot{\gamma}_{ij}(\tau)\dot{T}(\eta)\,\mathrm{d}\tau\,\mathrm{d}\eta \\
&\quad+\frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t}\frac{\partial}{\partial t}m(t-\tau,t-\eta)\dot{T}(\tau)\dot{T}(\eta)\,\mathrm{d}\tau\,\mathrm{d}\eta \\
&\quad+\frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t}\frac{\partial}{\partial t}H_{\alpha i\beta j}(t-\tau,t-\eta)\dot{\kappa}_{\alpha i}(\tau)\dot{\kappa}_{\beta j}(\eta)\,\mathrm{d}\tau\,\mathrm{d}\eta \\
&\quad+\int_{-\infty}^{t}\int_{-\infty}^{t}F_{\alpha ijk}(t-\tau,t-\eta)\dot{\kappa}_{\alpha i}(\tau)\dot{\gamma}_{jk}(\eta)\,\mathrm{d}\tau\,\mathrm{d}\eta \\
&\quad+\int_{-\infty}^{t}\int_{-\infty}^{t}\frac{\partial}{\partial t}\Gamma_{\alpha i}(t-\tau,t-\eta)\dot{\kappa}_{\alpha i}(\tau)\dot{T}(\eta)\,\mathrm{d}\tau\,\mathrm{d}\eta,
\end{aligned}
\tag{3.5}
$$

and we have made use of the symmetry conditions (3.2).

The inequality (3.4) must hold for arbitrary continuous histories of $\dot{\gamma}_{ij}(\tau)$, $\dot{\kappa}_{i\alpha}(\tau)$, $\dot{T}(\tau)$. The integral operators in (3.4) depend smoothly upon the histories, and thus changing the histories in the neighborhood of the present time $\tau = t$ produces only a small change in the value of the integrals. If the present rates could be assigned in a completely arbitrary manner without changing the values of the coefficients of the present rates, then, clearly, in order that the inequality be satisfied it would be necessary that the coefficients vanish identically. It can be shown that this is also true in the present case. The formal argument is similar to that of Coleman [6]; for the interested reader the details are given in an Appendix.

The vanishing of the coefficients then yields

$$\rho S = -\lambda(0) - \int_{-\infty}^{t} m(0, t-\tau)\dot{T}(\tau)\,d\tau - \int_{-\infty}^{t} \Phi_{ij}(t-\tau, 0)\dot{\gamma}_{ij}(\tau)\,d\tau - \int_{-\infty}^{t} \Gamma_{\alpha i}(t-\tau, 0)\dot{\kappa}_{\alpha i}(\tau)\,d\tau, \tag{3.6}$$

$$\begin{aligned}\pi_{\alpha\beta} = 2\bigg[D_{\alpha\beta}(0) &+ \int_{-\infty}^{t} G_{\alpha\beta kl}(0, t-\tau)\dot{\gamma}_{kl}(\tau)\,d\tau + \int_{-\infty}^{t} \Phi_{\alpha\beta}(0, t-\tau)\dot{T}(\tau)\,d\tau \\ &+ \int_{-\infty}^{t} F_{\gamma i\alpha\beta}(t-\tau, 0)\dot{\kappa}_{\gamma i}(\tau)\,d\tau \bigg],\end{aligned} \tag{3.7}$$

$$\begin{aligned}n_k = 2\bigg[D_{k3}(0) &+ \int_{-\infty}^{t} G_{k3mn}(0, t-\tau)\dot{\gamma}_{mn}(\tau)\,d\tau + \int_{-\infty}^{t} \Phi_{k3}(0, t-\tau)\dot{T}(\tau)\,d\tau \\ &+ \int_{-\infty}^{t} F_{\alpha ik3}(t-\tau, 0)\dot{\kappa}_{\alpha i}(\tau)\,d\tau \bigg],\end{aligned} \tag{3.8}$$

$$\begin{aligned}p_{\alpha i} = H_{\alpha i}(0) &+ \int_{-\infty}^{t} H_{\alpha i\beta j}(0, t-\tau)\dot{\kappa}_{\beta j}(\tau)\,d\tau + \int_{-\infty}^{t} F_{\alpha ijk}(0, t-\tau)\dot{\gamma}_{jk}(\tau)\,d\tau \\ &+ \int_{-\infty}^{t} \Gamma_{\alpha i}(0, t-\tau)\dot{T}(\tau)\,d\tau.\end{aligned} \tag{3.9}$$

The inequality (3.4) then reduces to

$$\begin{aligned}&-\int_{-\infty}^{t} \frac{\partial}{\partial t} D_{ij}(t-\tau)\dot{\gamma}_{ij}(\tau)\,d\tau - \int_{-\infty}^{t} \frac{\partial}{\partial t}\lambda(t-\tau)\,\dot{T}(\tau)\,d\tau \\ &-\int_{-\infty}^{t} \frac{\partial}{\partial t} H_{\alpha i}(t-\tau)\dot{\kappa}_{\alpha i}(\tau)\,d\tau - \Lambda - \frac{h}{T_0}\frac{\partial T}{\partial x} \geq 0.\end{aligned} \tag{3.10}$$

The first three terms in (3.10) are of first order in the rate histories, while the remaining terms are of second order. Hence, to satisfy (3.10), we must have

$$-\Lambda - \frac{h}{T_0}\frac{\partial T}{\partial x} \geq 0 \tag{3.11}$$

and

$$-\int_{-\infty}^{t} \frac{\partial}{\partial t} D_{ij}(t-\tau)\dot{\gamma}_{ij}(\tau)\,d\tau - \int_{-\infty}^{t} \frac{\partial}{\partial t}\lambda(t-\tau)\dot{T}(\tau)\,d\tau - \int_{-\infty}^{t} \frac{\partial}{\partial t} H_{\alpha i}(t-\tau)\dot{\kappa}_{\alpha i}(\tau)\,d\tau \geq 0. \tag{3.12}$$

As $\dot{\gamma}_{ij}(\tau)$, $\dot{T}(\tau)$ and $\dot{\kappa}_{\alpha i}(\tau)$ can be chosen arbitrarily, we must have

$$\frac{\partial}{\partial t}D_{ij}(t) = 0, \qquad \frac{\partial}{\partial t}\lambda(t) = 0, \qquad \frac{\partial}{\partial t}H_{\alpha i}(t) = 0. \tag{3.13}$$

Since (3.11) must hold for homogeneous temperature distributions, we also have

$$-\Lambda \geq 0. \tag{3.14}$$

From (3.11), we see that the familiar condition

$$-h\frac{\partial T}{\partial x} \geq 0 \tag{3.15}$$

does not necessarily follow from the entropy production inequality for a viscoelastic rod. However, to satisfy (3.11) it is sufficient that (3.15) be valid. If we restrict our attention to a class of materials for which (3.15) holds, it is easily seen that the constitutive equation (3.3) must reduce to

$$h = -\kappa\frac{\partial T}{\partial x}, \tag{3.16}$$

where κ (>0) is a constant. The arguments leading to (3.6)–(3.9) and (3.13) and (3.16) are very similar to those used by Christiansen and Naghdi [8] in their work on linear viscoelastic solids.

Substituting from (3.1) and (3.6–9) into the energy equation (2.17) and neglecting second order terms, we obtain a linearized energy equation

$$-\rho T_0 S + \rho r - \frac{\partial h}{\partial x} = 0 \tag{3.17}$$

which can be used to determine the temperature.

4. SYMMETRIES

We restrict our attention now to a rod whose Helmholtz function is invariant under the transformations:

$$X \to \pm X, \qquad \mathbf{A}_1 \to \pm\mathbf{A}_1, \qquad \mathbf{A}_2 \to \pm\mathbf{A}_2, \tag{4.1}$$

where we may take any combination of + and −. It is a straightforward, but tedious, calculation to find the form of the kernel functions under these restrictions. We present here only the final results. The free energy A is now given by

$$\begin{aligned}
\rho_0 A = \rho_0 A_0 - \rho_0 S_0 T + \int_{-\infty}^{t}\int_{-\infty}^{t} &\{\tfrac{1}{2}g_1(,)\dot{\gamma}_{12}(\tau)\dot{\gamma}_{12}(\eta) + \tfrac{1}{2}g_2(,)\dot{\gamma}_{23}(\tau)\dot{\gamma}_{23}(\eta) + \tfrac{1}{2}g_3(,)\dot{\gamma}_{13}(\tau)\dot{\gamma}_{13}(\eta) \\
&+ \tfrac{1}{4}g_4(,)\dot{\gamma}_{11}(\tau)\dot{\gamma}_{11}(\eta) + \tfrac{1}{4}g_5(,)\dot{\gamma}_{22}(\tau)\dot{\gamma}_{22}(\eta) + \tfrac{1}{4}g_6(,)\dot{\gamma}_{33}(\tau)\dot{\gamma}_{33}(\eta) + \tfrac{1}{4}g_7(,)[\dot{\gamma}_{11}(\tau)\dot{\gamma}_{22}(\eta) \\
&+ \dot{\gamma}_{22}(\tau)\dot{\gamma}_{11}(\eta)] + \tfrac{1}{4}g_8(,)[\dot{\gamma}_{22}(\tau)\dot{\gamma}_{33}(\eta) + \dot{\gamma}_{33}(\tau)\dot{\gamma}_{22}(\eta)] + \tfrac{1}{4}g_9(,)[\dot{\gamma}_{11}(\tau)\dot{\gamma}_{33}(\eta) \\
&+ \dot{\gamma}_{33}(\tau)\dot{\gamma}_{11}(\eta)] + [\tfrac{1}{2}\varphi_1(,)\dot{\gamma}_{11}(\tau) + \tfrac{1}{2}\varphi_2(,)\dot{\gamma}_{22}(\tau) + \tfrac{1}{2}\varphi_3(,)\dot{\gamma}_{33}(\tau)]\dot{T}(\eta) + \tfrac{1}{2}m(,)\dot{T}(\tau)\dot{T}(\eta) \qquad (4.2) \\
&+ \tfrac{1}{2}h_1(,)\dot{\kappa}_{11}(\tau)\dot{\kappa}_{11}(\eta) + \tfrac{1}{2}h_2(,)[\dot{\kappa}_{11}(\tau)\dot{\kappa}_{22}(\eta) + \dot{\kappa}_{22}(\tau)\dot{\kappa}_{11}(\eta)] + \tfrac{1}{2}h_3(,)\dot{\kappa}_{22}(\tau)\dot{\kappa}_{22}(\eta) \\
&+ \tfrac{1}{2}h_4(,)\dot{\kappa}_{12}(\tau)\dot{\kappa}_{12}(\eta) + \tfrac{1}{2}h_5(,)[\dot{\kappa}_{12}(\tau)\dot{\kappa}_{21}(\eta) + \dot{\kappa}_{21}(\tau)\dot{\kappa}_{12}(\eta)] + \tfrac{1}{2}h_6(,)\dot{\kappa}_{21}(\tau)\dot{\kappa}_{21}(\eta) \\
&+ \tfrac{1}{2}h_7(,)\dot{\kappa}_{13}(\tau)\dot{\kappa}_{13}(\eta) + \tfrac{1}{2}h_8(,)\dot{\kappa}_{23}(\tau)\dot{\kappa}_{23}(\eta)\}\, d\tau\, d\eta,
\end{aligned}$$

where we have used the notation

$$f(,) \equiv f(t-\tau, t-\eta),$$

and we have set $-\lambda(0) = \rho S_0$ and assumed that $D_{ij}(0) = 0$, i.e. the rod is free of initial stresses.

The constitutive equations (3.6), (3.7), (3.8) and (3.9) become

$$\begin{aligned} n_1 &= \int_{-\infty}^{t} g_3(0, t-\tau)\dot{\gamma}_{13}(\tau)\,\mathrm{d}\tau, \\ n_2 &= \int_{-\infty}^{t} g_2(0, t-\tau)\dot{\gamma}_{23}(\tau)\,\mathrm{d}\tau, \\ n_3 &= \int_{-\infty}^{t} [g_6(0, t-\tau)\dot{\gamma}_{33}(\tau) + g_8(0, t-\tau)\dot{\gamma}_{22}(\tau) + g_9(0, t-\tau)\dot{\gamma}_{11}(\tau)]\,\mathrm{d}\tau \\ &\quad + \int_{-\infty}^{t} \varphi_3(0, t-\tau)\dot{T}(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{4.3}$$

$$\begin{aligned} m_1 &= \int_{-\infty}^{t} h_8(0, t-\tau)\dot{\kappa}_{23}(\tau)\,\mathrm{d}\tau, \\ m_2 &= -\int_{-\infty}^{t} h_7(0, t-\tau)\dot{\kappa}_{13}(\tau)\,\mathrm{d}\tau, \\ m_3 &= \int_{-\infty}^{t} [(h_4 - h_5)(0, t-\tau)\dot{\kappa}_{12}(\tau) - (h_6 - h_5)(0, t-\tau)\dot{\kappa}_{21}(\tau)]\,\mathrm{d}\tau, \end{aligned} \tag{4.4}$$

$$\begin{aligned} p_{11} &= \int_{-\infty}^{t} [h_1(0, t-\tau)\dot{\kappa}_{11}(\tau) + h_2(0, t-\tau)\dot{\kappa}_{22}(\tau)]\,\mathrm{d}\tau, \\ p_{22} &= \int_{-\infty}^{t} [h_3(0, t-\tau)\dot{\kappa}_{22}(\tau) + h_2(0, t-\tau)\dot{\kappa}_{11}(\tau)]\,\mathrm{d}\tau, \\ p_{12} + p_{21} &= \int_{-\infty}^{t} [(h_4 + h_5)(0, t-\tau)\dot{\kappa}_{12}(\tau) + (h_6 + h_5)(0, t-\tau)\dot{\kappa}_{21}(\tau)]\,\mathrm{d}\tau, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \pi_{11} &= \int_{-\infty}^{t} [g_4(0, t-\tau)\dot{\gamma}_{11}(\tau) + g_7(0, t-\tau)\dot{\gamma}_{22}(\tau) + g_9(0, t-\tau)\dot{\gamma}_{33}(\tau)]\,\mathrm{d}\tau \\ &\quad + \int_{-\infty}^{t} \varphi_1(0, t-\tau)\dot{T}(\tau)\,\mathrm{d}\tau, \\ \pi_{22} &= \int_{-\infty}^{t} [g_5(0, t-\tau)\dot{\gamma}_{22}(\tau) + g_7(0, t-\tau)\dot{\gamma}_{11}(\tau) + g_8(0, t-\tau)\dot{\gamma}_{33}(\tau)]\,\mathrm{d}\tau \\ &\quad + \int_{-\infty}^{t} \varphi_2(0, t-\tau)\dot{T}(\tau)\,\mathrm{d}\tau, \\ \pi_{12} &= \pi_{21} = \int_{-\infty}^{t} g_1(0, t-\tau)\dot{\gamma}_{12}(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{4.6}$$

and

$$\rho S' = \rho S_0 - \int_{-\infty}^{t} m(0, t-\tau)\dot{T}(\tau)\,\mathrm{d}\tau - \frac{1}{2}\int_{-\infty}^{t} [\varphi_1(t-\tau, 0)\dot{\gamma}_{11}(\tau) + \varphi_2(t-\tau, 0)\dot{\gamma}_{22}(\tau) + \varphi_3(t-\tau, 0)\dot{\gamma}_{33}(\tau)]\,\mathrm{d}\tau. \tag{4.7}$$

The quantity Λ is now given by

$$\begin{aligned}\Lambda = \int_{-\infty}^{t}\int_{-\infty}^{t} \Bigg\{ & \frac{1}{2}\frac{\partial}{\partial t}g_1(,)\dot{\gamma}_{12}(\tau)\dot{\gamma}_{12}(\eta) + \frac{1}{2}\frac{\partial}{\partial t}g_2(,)\dot{\gamma}_{23}(\tau)\dot{\gamma}_{23}(\eta) + \frac{1}{2}\frac{\partial}{\partial t}g_3(,)\dot{\gamma}_{13}(\tau)\dot{\gamma}_{23}(\eta) \\ & + \frac{1}{4}\frac{\partial}{\partial t}g_2(,)\dot{\gamma}_{11}(\tau)\dot{\gamma}_{11}(\eta) + \frac{1}{4}\frac{\partial}{\partial t}g_5(,)\dot{\gamma}_{22}(\tau)\dot{\gamma}_{22}(\eta) + \frac{1}{4}\frac{\partial}{\partial t}g_6(,)\dot{\gamma}_{33}(\tau)\dot{\gamma}_{33}(\eta) \\ & + \frac{1}{2}\frac{\partial}{\partial t}g_7(,)\dot{\gamma}_{11}(\tau)\dot{\gamma}_{22}(\eta) + \frac{1}{2}\frac{\partial}{\partial t}g_8(,)\dot{\gamma}_{22}(\tau)\dot{\gamma}_{33}(\eta) + \frac{1}{2}\frac{\partial}{\partial t}g_9(,)\dot{\gamma}_{11}(\tau)\dot{\gamma}_{33}(\eta) \\ & + \left[\frac{1}{2}\frac{\partial}{\partial t}\varphi_1(,)\dot{\gamma}_{11}(\tau) + \frac{1}{2}\frac{\partial}{\partial t}\varphi_2(,)\dot{\gamma}_{22}(\tau) + \frac{1}{2}\frac{\partial}{\partial t}\varphi_3(,)\dot{\gamma}_{33}(\tau)\right]\dot{T}(\eta) \\ & + \frac{1}{2}\frac{\partial}{\partial t}m(,)\dot{T}(\tau)\dot{T}(\eta) + \frac{1}{2}\frac{\partial}{\partial t}h_1(,)\dot{\kappa}_{11}(\tau)\dot{\kappa}_{11}(\eta) + \frac{\partial}{\partial t}h_2(,)\dot{\kappa}_{11}(\tau)\dot{\kappa}_{22}(\eta) \\ & + \frac{1}{2}\frac{\partial}{\partial t}h_3(,)\dot{\kappa}_{22}(\tau)\dot{\kappa}_{22}(\eta) + \frac{1}{2}\frac{\partial}{\partial t}h_4(,)\dot{\kappa}_{12}(\tau)\dot{\kappa}_{12}(\eta) + \frac{\partial}{\partial t}h_5(,)\dot{\kappa}_{12}(\tau)\dot{\kappa}_{21}(\eta) \\ & + \frac{1}{2}\frac{\partial}{\partial t}h_6(,)\dot{\kappa}_{21}(\tau)\dot{\kappa}_{21}(\eta) + \frac{1}{2}\frac{\partial}{\partial t}h_7(,)\dot{\kappa}_{13}(\tau)\dot{\kappa}_{13}(\eta) + \frac{1}{2}\frac{\partial}{\partial t}h_8(,)\dot{\kappa}_{23}(\tau)\dot{\kappa}_{23}(\eta)\Bigg\}\,\mathrm{d}\tau\,\mathrm{d}\eta.\end{aligned}$$

Inspection of the equations of motion (2.9), (2.14) and (2.15), the energy equation (2.18) and the constitutive equations (4.3–7) shows that they can be separated into four distinct groups: two governing flexure, one governing torsion and one governing longitudinal motions. Thermal effects and the energy equation need only be considered with the equations governing longitudinal motions.

5. THE FLEXURE EQUATIONS

The two sets of equations governing the flexure of the rod are

$$\frac{\partial n_1}{\partial x} + \rho f_1 = \rho\frac{\partial^2 u_1}{\partial t^2}, \qquad \frac{\partial m_2}{\partial x} + n_1 - \rho l_{13} + \rho\alpha_1\frac{\partial^2 b_{13}}{\partial t^2} = 0$$

$$n_1 = \int_{-\infty}^{t} g_3(0, t-\tau)\dot{\gamma}_{13}(\tau)\,\mathrm{d}\tau, \qquad m_2 = -\int_{-\infty}^{t} h_7(0, t-\tau)\dot{\kappa}_{13}(\tau)\,\mathrm{d}\tau, \tag{5.1}$$

$$\gamma_{13} = b_{13} + b_{31}, \qquad \kappa_{13} = \frac{\partial b_{13}}{\partial x}, \qquad b_{31} = \frac{\partial u_1}{\partial x},$$

and

$$\frac{\partial n_2}{\partial x}+\rho f_2 = \rho\frac{\partial^2 u_2}{\partial t^2}, \qquad \frac{\partial m_1}{\partial x}-n_2+\rho l_{23}-\rho\alpha_2\frac{\partial^2 b_{23}}{\partial t^2} = 0,$$

$$n_2 = \int_{-\infty}^{t} g_2(0, t-\tau)\dot{\gamma}_{23}(\tau)\,d\tau, \qquad m_1 = \int_{-\infty}^{t} h_8(0, \tau-\tau)\dot{\kappa}_{23}(\tau)\,d\tau, \tag{5.2}$$

$$\gamma_{23} = b_{23}+b_{32}, \qquad \kappa_{23} = \frac{\partial b_{23}}{\partial x}, \qquad b_{32} = \frac{\partial u_2}{\partial x}.$$

As a simple application of these equations we may discuss the propagation of flexural waves along an infinite rod.† To simplify our discussion we introduce the notation

$$G(s) = g_2(0, s), \qquad H(s) = h_8(0, s). \tag{5.3}$$

Assuming that the waves are of the forms

$$\begin{aligned} u_2 &= U\,e^{-\mu x}\cos\omega\left(t-\frac{x}{c}\right), \\ b_{23} &= B_1\,e^{-\mu x}\cos\omega\left(t-\frac{x}{c}\right)+B_2\,e^{-\mu x}\sin\omega\left(t-\frac{x}{c}\right), \end{aligned} \tag{5.4}$$

on substitution of (5.4) into $(5.2)_{5,6,7}$, we find

$$\begin{aligned} n_2 &= \left[-\omega(B_1-\mu U)G_c+\omega\left(B_2+\frac{\omega}{c}U\right)G_s\right]e^{-\mu x}\sin\omega\left(t-\frac{x}{c}\right) \\ &\quad+\left[\omega(B_1-\mu U)G_s+\omega\left(B_2+\frac{\omega}{c}U\right)G_c\right]e^{-\mu x}\cos\omega\left(t-\frac{x}{c}\right), \\ m_1 &= \left[\omega\left(\mu B_1+\frac{\omega}{c}B_2\right)H_c+\omega\left(-\mu B_2+\frac{\omega}{c}B_1\right)H_s\right]e^{-\mu x}\sin\omega\left(t-\frac{x}{c}\right) \\ &\quad+\left[-\omega\left(\mu B_1+\frac{\omega}{c}B_2\right)H_c+\omega\left(-\mu B_2+\frac{\omega}{c}B_1\right)H_s\right]e^{-\mu x}\cos\omega\left(t-\frac{x}{c}\right), \end{aligned} \tag{5.5}$$

where

$$G_s(\omega) = \int_0^\infty G(u)\sin\omega u\,du, \qquad G_c(\omega) = \int_0^\infty G(u)\cos\omega u\,du, \tag{5.6}$$

and $H_s(\omega)$ and $H_c(\omega)$ are similarly defined as the Fourier sine and cosine transforms of $H(u)$. Using (5.5), (5.4) and the equations of motion $(5.2)_{1,2}$, we obtain a set of homogeneous

† Throughout this Section, we restrict our attentions to the case when $\alpha_1, \alpha_2, \rho, g_1, g_2$, etc. are independent of z.

equations to determine U, B_1 and B_2

$$\left[-2\mu\frac{\omega}{c}G_c+\left(\mu^2-\frac{\omega^2}{c^2}\right)G_s+\rho\omega\right]U+\left[-\mu G_s+\frac{\omega}{c}G_c\right]B_1-\left[\mu G_c+\frac{\omega}{c}G_s\right]B_2=0,$$

$$\left[2\mu\frac{\omega}{c}G_s+\left(\mu^2-\frac{\omega^2}{c^2}\right)G_c\right]U-\left[\mu G_c+\frac{\omega}{c}G_s\right]B_1+\left[-\frac{\omega}{c}G_c+\mu G_s\right]B_2=0,$$

$$\left[\mu G_s-\frac{\omega}{c}G_c\right]U+\left[\mu^2H_s-2\mu\frac{\omega}{c}H_c-\frac{\omega^2}{c^2}H_s-G_s+\rho\alpha_2\omega\right]B_1$$
$$+\left[2\mu\frac{\omega}{c}H_s+\mu^2H_c-\frac{\omega^2}{c^2}H_c-G_c\right]B_2=0, \tag{5.7}$$

$$\left[\mu G_s-\frac{\omega}{c}G_c\right]U+\left[-2\mu\frac{\omega}{c}H_s+\frac{\omega^2}{c}H_c-\mu^2H_c-G_s\right]B_1$$
$$+\left[-\frac{\omega^2}{c^2}H_s+\mu^2H_s-2\mu\frac{\omega}{c}H_c-G_c+\rho\alpha_2\omega\right]B_2=0.$$

These equations will determine μ, c and the direction of the vector $\{U, B_1, B_2\}$ as functions of the frequency ω. The solution for an arbitrary value of the frequency is quite complex, and we will restrict our attention to the asymptotic behavior of the solution as the frequency becomes very large. If $G(u)$ and $H(u)$ are sufficiently smooth, it is well known (see, e.g. Lighthill [7]) that

$$G_s(\omega)\sim\frac{G(0)}{\omega}-\frac{\ddot{G}(0)}{\omega^3}+\ldots,\qquad G_c(\omega)\sim-\frac{\dot{G}(0)}{\omega^2}+\frac{\dddot{G}(0)}{\omega^4}+\ldots \tag{5.8}$$

with similar expansions for $H_s(\omega)$ and $H_c(\omega)$. We assume that U, B_1, B_2, μ and c have asymptotic expansions of the form

$$U\sim u_0+\frac{u_1}{\omega}+\ldots,\qquad B_1\sim b_{10}+\frac{b_{11}}{\omega}+\ldots,$$
$$B_2\sim b_{20}+\frac{b_{21}}{\omega}+\ldots,\qquad \mu\sim\mu_0+\frac{\mu_1}{\omega}+\ldots,\qquad c\sim c_0+\frac{c_1}{\omega}+\ldots. \tag{5.9}$$

Substituting (5.8) and (5.9) into (5.7), and then investigating the conditions that non-trivial solutions exist, we find two modes of propagation for flexural waves. In the first,

$$c_0^2=\frac{G(0)}{\rho},\qquad \mu_0=-\frac{\dot{G}(0)}{2c_0G(0)},\qquad \mu_0\neq 0,\qquad b_{10}=b_{20}=0. \tag{5.10}$$

while in the second

$$c_0^2=\frac{H(0)}{\rho\alpha_2},\qquad \mu_0=-\frac{\dot{H}(0)}{2c_0H(0)},\qquad u_0=0,\qquad b_{10}=0,\qquad b_{20}\neq 0. \tag{5.11}$$

These relations generalize well known results for extensional waves in a one dimensional visco-elastic medium (see, e.g. Hunter [4]).

Our developments here rest on the initial postulates of a balance of energy and an entropy inequality. For a curve with assigned directors embedded in a Euclidean 3-space the resulting equations in Sections 2 and 3 are exact and aside from linearization involve no approximations. However, the equations given here could be obtained from a three dimensional theory of rods after suitable approximations are made. A discussion of the kind of approximation necessary is given in [3].

To one familiar with classical theories of rods, the form of our equations is quite suggestive, and we will present here a few rather intuitive ideas concerning the connection between the two theories.

A rod as a three dimensional body may be regarded as the Cartesian product of its line of centroids and its cross section. Because a rod is a "thin" body, we can constrain the cross section to undergo only homogeneous deformations. The homogeneous deformation of the cross section is then determined by the motion of two independent vectors. Thus in the model of the Cosserat curve, the two directors can serve to describe the homogeneous deformations of the cross section. Since any two independent vectors uniquely determine a homogeneous deformation, we may choose to take the directors along the principal axes of inertia of the cross section.

These ideas will be sufficient to permit us to compare the results of the Cosserat theory with solutions from the theory of linear viscoelasticity for an isotropic three dimensional rod. We consider first the case of isothermal, quasi-static pure flexure. Using the usual quasi-static "correspondence principle" between elastic and viscoelastic solutions, the solution of the viscoelastic problem in the Laplace transform plane is easily obtained from the corresponding elastic solution. We consider a straight, isotropic prismatic rod and introduce a Cartesian coordinate system (x, y, z) where the z axis coincides with the line of centroids of the cross section, and the x and y axes are taken along the principal axes of the cross section. Consider now the problem of pure flexure of such a body. If $\bar{\sigma}_{ij}$ denotes the transform of the stress tensor, the only non-zero component of the stress is

$$\bar{\sigma}_{33} = \alpha x + \beta y, \tag{5.12}$$

where α and β are constants. The three dimensional displacement vector is denoted by $\mathbf{u}^*$ and its transform is given by

$$\begin{aligned}
\bar{u}_1^* &= -\frac{\bar{\nu}}{s\bar{E}}\left(\frac{1}{2}\alpha x^2 + \beta xy - \frac{1}{2}\alpha y^2\right) - \frac{\alpha}{2s\bar{E}}z^2, \\
\bar{u}_2^* &= -\frac{\bar{\nu}}{s\bar{E}}\left(\alpha xy + \frac{1}{2}\beta y^2 - \frac{1}{2}\beta x^2\right) - \frac{\beta}{2s\bar{E}}z^2, \\
\bar{u}_3^* &= \frac{1}{s\bar{E}}(\alpha x + \beta y)z,
\end{aligned} \tag{5.13}$$

where $\bar{\nu}$ and $\bar{E}$ are defined in terms of $G_1(t)$, the relaxation function in shear, and $G_2(t)$, the relaxation function in isotropic compression, by

$$\frac{1}{\bar{E}} = \frac{2\bar{G}_2 + \bar{G}_1}{3\bar{G}_1\bar{G}_2}, \qquad \bar{\nu} = \frac{\bar{G}_2 - \bar{G}_1}{2\bar{G}_2 + \bar{G}_1}. \tag{5.14}$$

The transforms of the moments acting across any section are then

$$\bar{m}_1 = \beta I_{xx} = -s\bar{E}I_{xx}\frac{\partial^2 \bar{u}_2^*}{\partial z^2}\bigg|_{x,y=0}, \qquad \bar{m}_2 = -\alpha I_{yy} = s\bar{E}I_{yy}\frac{\partial^2 \bar{u}_1^*}{\partial z^2}\bigg|_{x,y=0}, \tag{5.15}$$

where I_{xx} and I_{yy} are moments of inertia about the x and y axes respectively.

Taking the transforms of (5.1) and (5.2)† we easily obtain the equations

$$\bar{n}_1 = 0, \qquad \bar{m}_2 = -s\bar{h}_7\frac{\partial \bar{b}_{13}}{\partial z}, \qquad \bar{b}_{13}+\bar{b}_{31} = 0, \qquad \bar{b}_{31} = \frac{\partial \bar{u}_1}{\partial z} \tag{5.16}$$

and

$$\bar{n}_2 = 0, \qquad \bar{m}_1 = s\bar{h}_8\frac{\partial \bar{b}_{23}}{\partial z}, \qquad \bar{b}_{23}+\bar{b}_{32} = 0, \qquad \bar{b}_{32} = \frac{\partial \bar{u}_2}{\partial z}. \tag{5.17}$$

Thus

$$\bar{m}_1 = -s\bar{h}_8\frac{\partial^2 \bar{u}_2}{\partial z^2}, \qquad \bar{m}_2 = s\bar{h}_7\frac{\partial^2 \bar{u}_1}{\partial z^2}, \tag{5.18}$$

and comparing these results with (5.15), we obtain

$$\bar{h}_8 = \bar{E}I_{xx}, \qquad \bar{h}_7 = \bar{E}I_{yy}. \tag{5.19}$$

6. TORSION

The equations governing torsional motion of the rod are

$$\begin{aligned}
&\frac{\partial m_3}{\partial z}+\rho(l_{12}-l_{21}) = \rho\left(\alpha_1\frac{\partial^2 b_{12}}{\partial t^2}-\alpha_2\frac{\partial^2 b_{21}}{\partial t^2}\right),\\
&\frac{\partial(p_{12}+p_{21})}{\partial z}+\rho(l_{12}+l_{21})-\rho\left(\alpha_1\frac{\partial^2 b_{12}}{\partial t^2}+\alpha_2\frac{\partial^2 b_{21}}{\partial t^2}\right) = 2\pi_{12},\\
&m_3 = \int_{-\infty}^{t}[(h_4-h_5)(0,t-\tau)\dot{\kappa}_{12}(\tau)-(h_6-h_5)(0,t-\tau)\dot{\kappa}_{21}(\tau)]\,\mathrm{d}\tau,\\
&p_{12}+p_{21} = \int_{-\infty}^{t}[(h_4+h_5)(0,t-\tau)\dot{\kappa}_{12}(\tau)+(h_6+h_5)(0,t-\tau)\dot{\kappa}_{21}(\tau)]\,\mathrm{d}\tau,\\
&\pi_{12} = \int_{-\infty}^{t} g_1(0,t-\tau)\dot{\gamma}_{12}(\tau)\,\mathrm{d}\tau,\\
&\kappa_{12} = \frac{\partial b_{12}}{\partial x}, \qquad \kappa_{21} = \frac{\partial b_{21}}{\partial x}, \qquad \gamma_{12} = b_{12}+b_{21}.
\end{aligned} \tag{6.1}$$

† We assume $u_i(\tau)$ and $b_{ij}(\tau)$ vanish on $-\infty < \tau \leq 0$.

We restrict our attention here to the case when the free energy and kinetic energy are invariant under the transformations

$$\mathbf{A}_1 \to \mathbf{A}_2, \qquad \mathbf{A}_2 \to \mathbf{A}_1. \tag{6.2}$$

This restriction implies that

$$\alpha_1 = \alpha_2, \qquad h_4 = h_6. \tag{6.3}$$

Under these conditions, we seek solutions of (6.1) with

$$\gamma_{12} = 0, \qquad b_{12} = b_{12}^*(x)\,\mathrm{e}^{i\omega t}. \tag{6.4}$$

Then, if the assigned loads are assumed to vanish, equations (6.1) reduce to

$$\begin{aligned} \frac{\partial m_3}{\partial z} &= -\rho\alpha\omega^2 b_{12}^*\,\mathrm{e}^{i\omega t}, \\ m_3 &= 2\hat{H}_1 \frac{\mathrm{d}b_{12}^*}{\mathrm{d}z}\,\mathrm{e}^{i\omega t}, \end{aligned} \tag{6.5}$$

where

$$\begin{aligned} \alpha &= 2\alpha_1 - 2\alpha_2, \\ H_1(0,u) &= h_4(0,u) - h_5(0,u), \\ \hat{H}_1 &= i\omega \int_0^\infty H_1(0,u)\,\mathrm{e}^{-i\omega u}\,\mathrm{d}u, \\ &= i\omega \bar{H}_1(0, i\omega). \end{aligned} \tag{6.6}$$

Combining (6.5a, b), we obtain

$$\frac{\mathrm{d}^2 b_{12}^*}{\mathrm{d}z^2} + q^2 b_{12}^* = 0, \tag{6.7}$$

where

$$q^2 = \frac{\rho\alpha\omega^2}{2\hat{H}_1}. \tag{6.8}$$

If we take $b_{12}^*(0) = 0$, the solution of (6.7) is

$$b_{12}^* = b \sin qz. \tag{6.9}$$

The torque necessary to maintain this deformation is

$$m_3 = b\frac{\rho\alpha\omega^2}{q} \cos qz\,\mathrm{e}^{i\omega t}. \tag{6.10}$$

This result may be compared with Berry's [5] result for the torsional vibrations of a circular cylinder obtained from the three dimensional theory. With some changes in notation, Berry's expression for the torque necessary to maintain sinusoidal oscillation is

$$m_3 = b\frac{\rho a^2 \omega^2}{2\lambda} \cos \lambda z, \qquad \lambda^2 = \frac{2\rho\omega^2}{\pi a^2 [i\omega \bar{G}_1(i\omega)]}, \tag{6.11}$$

where a is the radius of the cylinder. The tangential displacement of points in the cross section is given by

$$u_\theta = br \sin qz\, e^{i\omega t}. \tag{6.12}$$

Thus, b_{12} can be interpreted as the rotation of a radial line element in the cross section. Equations (6.11) and (6.10) are equivalent if

$$\alpha = \frac{a^2}{2}, \qquad \bar{H}_1(i\omega) = \tfrac{1}{4} I_z \bar{G}_1(i\omega), \tag{6.13}$$

where I_z is the polar moment of inertia of the cross section.

7. LONGITUDINAL MOTIONS

In the theory employed in this paper, thermal effects only arise in the case of extensional motion of the rod. We restrict our attention to the case when the heat flux is given by (3.19). Then, the equations governing the motion are

$$\begin{aligned}
&\frac{\partial n_3}{\partial x} + \rho f_3 = \rho \frac{\partial^2 u_3}{\partial t^2},\\
&\pi_{11} = \frac{\partial p_{11}}{\partial x} + \rho\left(l_{11} - \alpha_1 \frac{\partial^2 b_{11}}{\partial t^2}\right),\\
&\pi_{22} = \frac{\partial p_{22}}{\partial x} + \rho\left(l_{22} - \alpha_2 \frac{\partial^2 b_{22}}{\partial t^2}\right),\\
&-\rho T_0 \dot{S} + \rho r - \frac{\partial h}{\partial x} = 0,\\
&n_3 = \int_{-\infty}^{t} [g_6(0, t-\tau)\dot{\gamma}_{33}(\tau) + g_8(0, t-\tau)\dot{\gamma}_{22}(\tau) + g_9(0, t-\tau)\dot{\gamma}_{11}(\tau)\\
&\qquad + \varphi_3(0, t-\tau)\dot{T}(\tau)]\, d\tau,\\
&p_{11} = \int_{-\infty}^{t} [h_1(0, t-\tau)\dot{\kappa}_{11}(\tau) + h_2(0, t-\tau)\dot{\kappa}_{22}(\tau)]\, d\tau,\\
&p_{22} = \int_{-\infty}^{t} [h_3(0, t-\tau)\dot{\kappa}_{22}(\tau) + h_2(0, t-\tau)\dot{\kappa}_{11}(\tau)]\, d\tau,\\
&\pi_{11} = \int_{-\infty}^{t} [g_4(0, t-\tau)\dot{\gamma}_{11}(\tau) + g_7(0, t-\tau)\dot{\gamma}_{22}(\tau) + g_9(0, t-\tau)\dot{\gamma}_{33}(\tau)\\
&\qquad + \varphi_1(0, t-\tau)\dot{T}(\tau)]\, d\tau,\\
&\pi_{22} = \int_{-\infty}^{t} [g_5(0, t-\tau)\dot{\gamma}_{22}(\tau) + g_7(0, t-\tau)\dot{\gamma}_{11}(\tau) + g_8(0, t-\tau)\dot{\gamma}_{33}(\tau)\\
&\qquad + \varphi_2(0, t-\tau)\dot{T}(\tau)]\, d\tau,\\
&\gamma_{11} = 2b_{11}, \quad \gamma_{22} = 2b_{22}, \quad \gamma_{33} = 2\frac{\partial u_3}{\partial x}, \quad \kappa_{11} = \frac{\partial b_{11}}{\partial x}, \quad \kappa_{22} = \frac{\partial b_{22}}{\partial x}
\end{aligned} \tag{7.1}$$

Acknowledgement—The results reported here were obtained in the course of research supported by the U.S. Office of Naval Research under Contracts Nonr-222(69) and N00014-67-A-0114-0021 with the University of California, Berkeley. The author wishes to thank Professor P. M. Naghdi for suggesting the topic and for many interesting and helpful discussions on the subject.

REFERENCES

[1] A. E. Green and N. Laws, *Proc. R. Soc.* **A293**, 145 (1960).
[2] A. E. Green, N. Laws and P. M. Naghdi, *Archs. rat. Mech. Analysis* **23**, 135 (1966).
[3] A. E. Green, N. Laws and P. M. Naghdi, *Proc. Camb. Phil. Soc.* **64**, 895 (1968).
[4] S. C. Hunter, *Prog. Solid Mech.*, **1**, 3 (1960).
[5] D. S. Berry, *J. Mech. Phys. Solids* **6**, 177 (1958).
[6] B. D. Coleman, *Archs. rat. Mech. Analysis* **17**, 1 (1964).
[7] M. J. Lighthill, *Fourier Analysis and Generalized Functions*. Cambridge University Press (1958).
[8] R. M. Christensen and P. M. Naghdi, *Acta Mech.* **3**, 1 (1967).

APPENDIX

The inequality (3.4) can be written in a more revealing form if we introduce the vectors

$$\begin{aligned} \boldsymbol{\eta} &= (\tfrac{1}{2}\dot{\gamma}_{ij}, \dot{\kappa}_{\alpha i}, \dot{T}), \\ \boldsymbol{\Gamma} &= (\pi_{\alpha\beta}, n_i, \dot{\kappa}_{\alpha i}, -\rho S'). \end{aligned} \tag{A1}$$

Then the constitutive hypothesis for n_i, $p_{\alpha i}$, $\pi_{\alpha\beta}$ and S' in terms of a matrix kernel function $\mathbf{P}(\tau)$ and a constant vector $\boldsymbol{\Gamma}_0$ is

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_0 + \int_{-\infty}^{t} \mathbf{P}(t-\tau)\boldsymbol{\eta}(\tau)\,\mathrm{d}\tau. \tag{A2}$$

By introducing appropriate matrix kernel functions $\mathbf{G}(s, \tau)$ and $\mathbf{H}(\tau)$ in (3.1) we can write $\rho\dot{A}$ in the form:

$$\begin{aligned} \rho\dot{A} = {} & \left[\mathbf{H}(0) + \int_{-\infty}^{t} \mathbf{G}(0, t-\tau)\boldsymbol{\eta}(\tau)\,\mathrm{d}\tau\right]\boldsymbol{\eta}(t) + \int_{-\infty}^{t} \frac{\partial}{\partial t}\mathbf{H}(t-\tau)\boldsymbol{\eta}(\tau)\,\mathrm{d}\tau \\ & + \int_{-\infty}^{t}\int_{-\infty}^{t} \frac{\partial}{\partial t}\mathbf{G}(t-\tau, t-s) + \int_{-\infty}^{t}\int_{-\infty}^{t} \frac{\partial}{\partial t}\mathbf{G}(t-\tau, t-s)\boldsymbol{\eta}(\tau)\boldsymbol{\eta}(s)\,\mathrm{d}\tau\,\mathrm{d}s. \end{aligned} \tag{A3}$$

The inequality (3.4) for processes with $(\partial T/\partial z)(t) = 0$ then becomes

$$\begin{aligned} & \left\{\boldsymbol{\Gamma} - \mathbf{H}(0) + \int_{-\infty}^{t} [\mathbf{P}(t-\tau) - \mathbf{G}(0, t-\tau)]\boldsymbol{\eta}(\tau)\,\mathrm{d}\tau\right\}\boldsymbol{\eta}(\tau) \\ & - \int_{-\infty}^{t} \frac{\partial}{\partial t}\mathbf{H}(t-\tau)\boldsymbol{\eta}(\tau)\,\mathrm{d}\tau - \int_{-\infty}^{t}\int_{-\infty}^{t} \frac{\partial}{\partial t}\mathbf{G}(t-\tau, t-s)\boldsymbol{\eta}(\tau)\boldsymbol{\eta}(s)\,\mathrm{d}\tau\,\mathrm{d}s \geq 0. \end{aligned} \tag{A4}$$

We will assume that $\boldsymbol{\eta}(\tau)$, $(\partial/\partial t)\mathbf{H}(\tau)$, $(\partial/\partial t)\mathbf{G}(\tau, s)$, and $(\partial/\partial s)\mathbf{G}(\tau, s)$ are continuous (these assumptions have already been used when differentiating under the integral sign to find $\rho\dot{A}$). Typical terms in (A4) are of the form

$$\int_{-\infty}^{t} \boldsymbol{\varphi}(t-\tau)\boldsymbol{\eta}(\tau)\,\mathrm{d}\tau, \qquad \int_{-\infty}^{t}\int_{-\infty}^{t} \boldsymbol{\Psi}(t-\tau, t-s)\boldsymbol{\eta}(\tau)\boldsymbol{\eta}(s)\,\mathrm{d}\tau\,\mathrm{d}s \tag{A5}$$

where $\boldsymbol{\varphi}$ and $\boldsymbol{\Psi}$ are continuous functions. If $\boldsymbol{\eta}(\tau)$ is some specified, continuous history and if $\boldsymbol{\alpha}$ is an arbitrary vector, then given any $\varepsilon > 0$ we can find a continuous history $\hat{\boldsymbol{\eta}}(\tau)$ such that $\hat{\boldsymbol{\eta}}(t) = \alpha$ and

$$\begin{gathered}\left\| \int_{-\infty}^{t} \boldsymbol{\varphi}(t-\tau)\boldsymbol{\eta}(\tau)\,\mathrm{d}\tau - \int_{-\infty}^{t} \boldsymbol{\varphi}(t-\tau)\hat{\boldsymbol{\eta}}(\tau)\,\mathrm{d}\tau \right\| < \varepsilon, \\ \left\| \int_{-\infty}^{t}\int_{-\infty}^{t} \boldsymbol{\Psi}(t-\tau, t-s)\boldsymbol{\eta}(\tau)\boldsymbol{\eta}(s)\,\mathrm{d}\tau\,\mathrm{d}s - \int_{-\infty}^{t}\int_{-\infty}^{t} \boldsymbol{\Psi}(t-\tau, t-s)\boldsymbol{\eta}(\tau)\hat{\boldsymbol{\eta}}(s)\,\mathrm{d}\tau\,\mathrm{d}s \right\| < \varepsilon.\end{gathered} \tag{A6}$$

The norm in (A6) is the usual vector or tensor norm; i.e. if $\mathbf{V}$ is a vector $\|\mathbf{V}\| = [\mathbf{V}\mathbf{V}]^{1/2}$, if A is a tensor, then $\|\mathbf{A}\| = \mathrm{tr}[\mathbf{A}\mathbf{A}^T]^{1/2}$.

For example, consider

$$\begin{aligned}\hat{\boldsymbol{\eta}}(\tau) &= \hat{\boldsymbol{\eta}}(\tau) \qquad -\infty < \tau \leq t-\delta \\ &= \boldsymbol{\eta}(t-\delta) + [\boldsymbol{\alpha} - \boldsymbol{\eta}(t-\delta)]\left[\frac{\tau - t + \delta}{\delta}\right]. \qquad t-\delta \leq \tau \leq t\end{aligned} \tag{A7}$$

Then

$$\begin{aligned}&\left\| \int_{-\infty}^{t} \boldsymbol{\varphi}(t-\tau)\boldsymbol{\eta}(\tau)\,\mathrm{d}\tau - \int_{-\infty}^{t} \boldsymbol{\varphi}(t-\tau)\hat{\boldsymbol{\eta}}(\tau)\,\mathrm{d}\tau \right\| \\ &= \int_{t-\delta}^{t} \boldsymbol{\varphi}(t-\tau)\left\{\boldsymbol{\eta}(\tau) - \boldsymbol{\eta}(t-\delta) - [\alpha - \boldsymbol{\eta}(t-\delta]\left[\frac{\tau - t + \delta}{\delta}\right]\right. \mathrm{d}\tau\| \leq \Phi(3\Gamma + \|\alpha\|)\delta,\end{aligned}$$

where $\Phi = \sup\|\boldsymbol{\varphi}(t-\tau)\|$ and $\Gamma = \sup\|\boldsymbol{\eta}(\tau)\|$ on some sufficiently large interval containing $(t-\delta, t)$. Similarly

$$\left| \int_{-\infty}^{t}\int_{-\infty}^{t} \boldsymbol{\psi}(t-\tau, t-s)\boldsymbol{\eta}(\tau)\boldsymbol{\eta}(s)\,\mathrm{d}\tau\,\mathrm{d}s \right| \leq \boldsymbol{\Psi}[\Gamma^2 + (2\Gamma + \|\alpha\|)^2]\delta^2, \tag{A9}$$

where $\Psi = \sup\|\boldsymbol{\psi}(t-\tau, t-s)\|$. By choosing δ sufficiently small we obtain our desired result.

Consider now the inequality (A4) for a given history $\boldsymbol{\eta}(\tau)$. Choose another history $\hat{\boldsymbol{\eta}}(\tau)$ with $\hat{\boldsymbol{\eta}}(t) = \boldsymbol{\alpha}, \boldsymbol{\alpha}$ being an arbitrary vector. This history must also satisfy the inequality so that

$$\begin{aligned}&\left\{\boldsymbol{\Gamma}_0 - \mathbf{H}(0) + \int_{-\infty}^{t} [\mathbf{P}(t-\tau) - G(0, t-\tau)]\hat{\boldsymbol{\eta}}(\tau)\,\mathrm{d}\tau\right\}\boldsymbol{\alpha} \\ &- \int_{-\infty}^{t} \frac{\partial}{\partial t}\mathbf{H}(t-\tau)\hat{\boldsymbol{\eta}}(\tau)\,\mathrm{d}\tau - \int_{-\infty}^{t}\int_{-\infty}^{t} \frac{\partial}{\partial t}\mathbf{G}(t-\tau, t-s)\hat{\boldsymbol{\eta}}(\tau)\hat{\boldsymbol{\eta}}(s)\,\mathrm{d}\tau\,\mathrm{d}s \geq 0.\end{aligned} \tag{A10}$$

Adding and subtracting terms in (A10) we obtain:

$$\left\{\int_{-\infty}^{t} [\mathbf{P}(t-\tau)-G(0,t-\tau)][\hat{\boldsymbol{\eta}}(t)-\boldsymbol{\eta}(\tau)]\,\mathrm{d}\tau\right\}\cdot\boldsymbol{\alpha}-\int_{-\infty}^{t}\frac{\partial}{\partial t}\mathbf{H}(t-\tau)[\hat{\boldsymbol{\eta}}(\tau)-\boldsymbol{\eta}(\tau)]\,\mathrm{d}\tau$$
$$-\int_{-\infty}^{t}\int_{-\infty}^{t}\frac{\partial}{\partial t}\mathbf{G}(t-\tau,t-s)[\hat{\boldsymbol{\eta}}(\tau)\hat{\boldsymbol{\eta}}(s)-\boldsymbol{\eta}(\tau)\boldsymbol{\eta}(s)]\,\mathrm{d}\tau\,\mathrm{d}s$$
$$+\left\{\boldsymbol{\Gamma}_0-\mathbf{H}(0)+\int_{-\infty}^{t}[\mathbf{P}(t-\tau)-G(0,t-\tau)]\boldsymbol{\eta}(\tau)\,\mathrm{d}\tau\right\}\cdot\boldsymbol{\alpha} \tag{A11}$$
$$-\int_{-\infty}^{t}\frac{\partial}{\partial t}\mathbf{H}(t-\tau)\boldsymbol{\eta}(\tau)\,\mathrm{d}\tau-\int_{-\infty}^{t}\int_{-\infty}^{t}\frac{\partial}{\partial t}\mathbf{G}(t-\tau,t-s)\boldsymbol{\eta}(\tau)\boldsymbol{\eta}(s)\,\mathrm{d}\tau\,\mathrm{d}s\geq 0.$$

However, using (A6) we see that by choosing $\hat{\boldsymbol{\eta}}(\tau)$ properly the first three terms in (A11) can be made arbitrarily small, and hence if the inequality is to hold we must have

$$\left\{\boldsymbol{\Gamma}_0-\mathbf{H}(0)+\int_{-\infty}^{t}[\mathbf{P}(t-\tau)-G(0,t-\tau)]\boldsymbol{\eta}(\tau)\,\mathrm{d}\tau\right\}\cdot\boldsymbol{\alpha}$$
$$+\int_{-\infty}^{t}\frac{\partial}{\partial t}\mathbf{H}(t-\tau)\boldsymbol{\eta}(\tau)\,\mathrm{d}\tau+\int_{-\infty}^{t}\int_{-\infty}^{t}\frac{\partial}{\partial t}G(t-\tau,t-s)\boldsymbol{\eta}(\tau)\boldsymbol{\eta}(s)\,\mathrm{d}\tau\,\mathrm{d}s\geq 0. \tag{A12}$$

Since $\boldsymbol{\alpha}$ is arbitrary and $\boldsymbol{\eta}(\tau)$ is any continuous history, (A12) implies that

$$\boldsymbol{\Gamma}_0=\mathbf{H}(0), \tag{A13}$$

and

$$\mathbf{P}(t-\tau)=\mathbf{G}(0,t-\tau). \tag{A14}$$

The inequality (A4) then reduces to

$$-\int_{-\infty}^{t}\frac{\partial}{\partial t}\mathbf{H}(t-\tau)\boldsymbol{\eta}(\tau)\,\mathrm{d}\tau-\int_{-\infty}^{t}\int_{-\infty}^{t}\frac{\partial}{\partial t}\mathbf{G}(t-\tau,t-s)\boldsymbol{\eta}(\tau)\boldsymbol{\eta}(s)\,\mathrm{d}\tau\,\mathrm{d}s\geq 0. \tag{A15}$$

Let $\boldsymbol{\eta}^*(\tau)=\beta\boldsymbol{\eta}(\tau)$ where β is an arbitrary scalar. The inequality (A15) must still be satisfied for this new history so that

$$-\beta\int_{-\infty}^{t}\frac{\partial}{\partial t}\mathbf{H}(t-\tau)\boldsymbol{\eta}(\tau)\,\mathrm{d}\tau-\beta^2\int_{-\infty}^{t}\int_{-\infty}^{t}\frac{\partial}{\partial t}\mathbf{G}(t-\tau,t-s)\boldsymbol{\eta}(\tau)\boldsymbol{\eta}(s)\,\mathrm{d}\tau\,\mathrm{d}s\geq 0. \tag{A16}$$

In order that (A16) be satisfied, we must have

$$\frac{\partial}{\partial t}\mathbf{H}(t-\tau)=0, \tag{A17}$$

and

$$\int_{-\infty}^{t}\int_{-\infty}^{t}\frac{\partial}{\partial t}\mathbf{G}(t-\tau,t-s)\boldsymbol{\eta}(\tau)\boldsymbol{\eta}(s)\,\mathrm{d}\tau\,\mathrm{d}s>0. \tag{A18}$$

Our original inequality (3.4) in view of (A13), (A14) and (A17) reduces to

$$-\int_{-\infty}^{t}\int_{-\infty}^{t}\frac{\partial}{\partial t}\mathbf{G}(t-\tau, t-s)\boldsymbol{\eta}(\tau)\boldsymbol{\eta}(s)\,\mathrm{d}\tau\,\mathrm{d}s-\frac{h}{T_0}\frac{\partial T}{\partial z}\geq 0. \tag{A19}$$

If we revert to our usual index notation, (A13) and (A14) imply (3.6), (3.7), (3.8), (3.9) and (3.10).

(*Received* 17 *February* 1969; *revised* 4 *June* 1969)

Абстракт—Настоящая работа основана на общей термодинамической теории континуума Коссера, развитой Грином и Лявсом. Даются, здесь, специфичные определяющие уравнения для линейного вязкоупругого материала. Когда форма свободной энергии ограничена некоторыми условиями симметрии, основные уравнения разделяются на четырые группы, две для изгиба, одна для кручения и одна для растяжения стержня. Термические эффекты выступают только в последней группе. Рассматривается распределение волны изгиба и кручения вдоль бесконечного стержня.